ON A SYSTEM OF PARASTROIDS*

By R. P. Stephens

Introduction. If three points t_i be taken on a circle, say the unit circle, and if a fourth point τ of the same circle be projected at a fixed inclination k^2 on the three lines joining the points t_i, then these three projections lie on a line. As τ varies, this line envelops a three-cusped hypocycloid; by varying the angle of projection, Dr. Converse obtained a system of hypocycloids which he has discussed in detail.† In an earlier paper,‡ Prof. Steggall chose his points t_i so as to form a regular inscribed triangle, but he extended his investigation in this very special case to n points forming the vertices of a regular inscribed polygon of n sides. It is my intention to extend Dr. Converse's general treatment for three points to n points and to discuss at length the case where $n = 4$.§ For this purpose the algebraic work is greatly simplified by expressing the line equation of a curve in conjugate coordinates‖ and the point equation by means of mapping from the unit circle.

1. The Wallace Lines. The conjugate equation of a line through the points t_1 and t_2 of the unit circle is

$$(1) \qquad x + t_1 t_2 y = t_1 + t_2.$$

The reflexion of a point τ of the unit circle in this line is

$$x = t_1 + t_2 - \frac{t_1 t_2}{\tau}.$$

If now we represent the symmetric functions of t_i (where $i = 1, 2, 3$) by σ_1, σ_2, and σ_3, then this last equation becomes

$$(2) \qquad x = \sigma_1 - t - \frac{\sigma_3}{t\tau},$$

* Parts of this article appeared in the Johns Hopkins University Circular, January, 1905 (New Series, No. 1); other parts were presented to the American Mathematical Society, Dec. 27, 1906.

† H. A. Converse, Annals of Mathematics, ser. 2, vol. 5 (1904), pp. 105–139.

‡ Steggall, *Proceedings of the Edinburgh Mathematical Society*, vol. 14 (1895–6), p. 131.

§ This extension of Dr. Converse's work to four points was suggested by Professor Bromwich.

‖ F. Morley, *Trans. Amer. Math. Soc.*, vol. 1 (1900), p. 97, also vol. 4 (1904), pp. 1–12. H. A. Converse, loc. cit.

which for t equal to t_1, t_2, or t_3 gives the reflexion of τ in the line through the other two t's; hence, for varying t, this is the equation of some locus which passes through the three reflexions. The conjugate of (2) is

$$\sigma_3 y = \sigma_2 - \frac{\sigma_3}{t} - t\tau.$$

On eliminating t between (2) and its conjugate there results

$$(W_3) \qquad \tau x - \sigma_3 y = \sigma_1 \tau - \sigma_2,$$

the conjugate equation of a line, *i. e.*, the Wallace* line of the three points t_i as to τ. For convenience, I shall designate this line by W_3, the subscript referring to the number of t's.

If now four points t_i be used, there will be four lines W_3, obtained from the points t_i three at a time. The reflexion of τ in these four lines will be of the form

$$\tau x = \sigma_1 \tau - \sigma_2 + \frac{\sigma_3}{\tau}.$$

If this be made symmetrical for four t's, that is, if the substitution

$$\sigma_1 = \sigma_1' - t,$$
$$\sigma_2 = \sigma_2' - t\sigma_1' + t^2,$$
$$\sigma_3 = \sigma_4'/t,$$

be made, this gives

$$\tau x = (\sigma_1 - t)\,\tau - \sigma_2 + \sigma_1 t - t^2 + \sigma_4/t\tau,$$

where the accent has been dropped from σ_i and they now refer to four t's.

On eliminating t between this equation and its conjugate, there results

$$\tau^2 x + \sigma_4 y = \sigma_1 \tau^2 - \sigma_2 \tau + \sigma_3 - \frac{\tau}{t^2}\,(t^4 - \sigma_1 t^3 + \sigma_2 t^2 - \sigma_3 t + \sigma_4),$$

or, since $$t^4 - \sigma_1 t^3 + \sigma_2 t^2 - \sigma_3 t + \sigma_4 = 0,$$

$$(W_4) \qquad \tau^2 x + \sigma_4 y = \sigma_1 t^2 - \sigma_2 t + \sigma_3,$$

* Jno. S. MacKay, "The Wallace Line and the Wallace Point," *Proceedings of the Edinburgh Math. Soc.*, vol. 9 (1890–91). M. Cantor, *Geschichte der Math.*, vol. 3, p. 542.

the equation of the Wallace line for four points t_i as to τ. In a similar manner, we obtain, in general,

$$(W_n) \qquad \tau^{n-2}x + (-1)^n\sigma_n y = \sigma_1\tau^{n-2} - \sigma_2\tau^{n-3} + \sigma_3\tau^{n-4} \cdots + (-1)^n\sigma_{n-1},$$

the equation of the Wallace line* for n points t_i.

2. Some Curves of Class n, arising from the n Lines W_{n-1}. With these Wallace lines at hand it is now easy to write down the projection of a point τ on them at any fixed inclination.

Consider the equation of the line

$$(1) \qquad x + t_1t_2y = t_1 + t_2;$$

then

$$x + k^2t_1t_2y = \tau + \frac{k^2t_1t_2}{\tau}$$

is a line which meets (1) at a fixed inclination † and passes through the point τ. These two lines intersect at the point

$$(k^2-1)x = k^2\left(t_1 + t_2 - \frac{t_1t_2}{\tau}\right) - \tau,$$

which is therefore the *projection* of the point τ on the line (1) at a fixed inclination.

On symmetrizing this equation for three t's as on p. 160 we obtain

$$(k^2-1)x = k^2\left(\sigma_1 - t - \frac{\sigma_3}{t\tau}\right) - \tau,$$

or, eliminating t between this and its conjugate, we find

$$(C_3) \qquad \tau^3 - [k^2\sigma_1 - (k^2-1)x]\,\tau^2 + [k^2\sigma_2 + k^2(k^2-1)\sigma_3y]\tau - \sigma_3k^4 = 0,$$

the equation of a line, namely, the line on which the three projections must lie; but, as τ varies, C_3 envelops the deltoid which formed the basis of Dr. Converse's article referred to above.

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* This derivation of the Wallace lines is due to Professor Morley and was given by him in a course of lectures in 1903–4.

† If θ is the angle which the second line makes with the first, then $k^2 = e^{2i\theta}$; if the second makes the supplementary angle to θ with the first

$$k_1^2 = e^{2i(\pi-\theta)} = e^{-2i\theta} = \frac{1}{k^2}.$$

In a similar manner if we start with n fixed points on the circle and project a point τ of the same circle on the n lines W_{n-1} arising from the fixed points taken $n-1$ at a time, we obtain

$$(C_n) \qquad \tau^n - [k^2\sigma_1 - (k^2-1)x]\,\tau^{n-1} + k^2\sigma_2\tau^{n-2} + \cdots - (-1)^n \left\{ [k^2\sigma_{n-1} + k^2(k^2-1)\,\sigma_n y]\,\tau - k^4\sigma_n \right\} = 0,$$

the equation of a line.

Therefore the n projections of a point of a circle on the n Wallace lines, arising from n points of the circle taken $n-1$ at a time, lie on a line.

By the method explained later in the particular case where $n = 4$, it is found that *the curve enveloped by C_n, as τ varies, is a curve of class n and order $2n-2$, and has, in general, n cusps whose n cusp-tangents touch a curve of class $n-2$.*

3. The Equation of the Parastroid. Let us consider more particularly the equation C_4, which, if the axis of reals be so chosen that $\sigma_4 = 1$, may be written

$$(3) \qquad \tau^4 - [k^2\sigma_1 - (k^2-1)x]\,\tau^3 + k^2\sigma_2\tau^2 - [k^2\sigma_3 + k^2(k^2-1)y]\,\tau + k^4 = 0.$$

For a fixed value of τ, this is the equation of a line, but as τ varies the line envelops a curve which is of *fourth class*, since from any given point there are obviously four tangents. Its map equation, obtained by dividing by τ and then differentiating with respect to τ, is

$$(4) \qquad 2\left(x - \frac{k^2\sigma_1}{k^2-1}\right) = \frac{k^2}{k^2-1}\left[\left(\frac{k^2}{\tau^3} - \frac{3\tau}{k^2}\right) - \frac{\sigma_2}{\tau}\right],$$

in which k is a constant turn and τ is the variable turn. Since a line cuts it in six points, the curve is of the *sixth degree*. The curve is readily shown to be the parallel to an astroid, that is, it is a *parastroid*.*

From both (3) and (4) it is evident that the center of the curve is

$$x_0 = \frac{k^2\sigma_1}{k^2-1}.$$

If, however, k is allowed to vary, we shall obtain a system of parastroids the locus of whose centers is

$$x = \frac{k^2\sigma_1}{k^2-1},$$

*G. Loria, *Spezielle ebene Kurven*, p. 651.

a straight line perpendicular to the stroke σ_1 at its mid-point. Therefore, *the centers of all the parastroids of the system, obtained by varying k, lie on the right line perpendicular to the stroke σ_1 at it mid-point* (see figure 1).

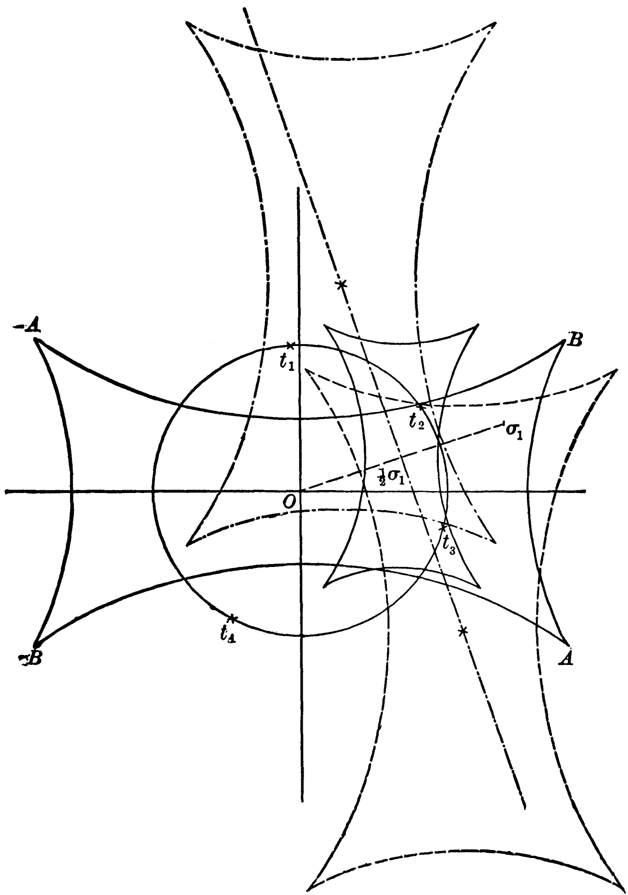

FIG. 1.

A similar theorem may be stated for the general case, that is, for the locus of the centers of the curves enveloped by the line C_n as k varies.

4. The Cusps of the Parastroid. The condition for cusps is obtained from (4) by equating $dx/d\tau$ to zero. It is

$$(5) \qquad \tau^4 - \tfrac{1}{3}\, \sigma_2^2 k^2 \tau^2 + k^4 = 0,$$

of which the roots are:

$$\tau_1 = ak, \qquad \tau_3 = -\,ak,$$

$$\tau_2 = k/a, \qquad \tau_4 = -\,k/a,$$

where $2a \equiv \sqrt{\tfrac{1}{3}\sigma_2 + 2} + \sqrt{\tfrac{1}{3}\sigma_2 - 2}$.

These four values of τ are turns (*i. e.*, restricted to the unit circle) for all values of σ_2 such that $\sigma_2 \gtreqless 6$, — a condition which, according to the formation of σ_2, is always satisfied. Putting these values of τ in (4), we find as the four cusps of that parastroid for which $k = k_1$:

$$x_1 = \frac{k_1 A + k_1^2\sigma_1}{k_1^2 - 1}, \qquad x_3 = \frac{-\,k_1 A + k_1^2\sigma_1}{k_1^2 - 1},$$

$$x_2 = \frac{k_1 B + k_1^2\sigma_1}{k_1^2 - 1}, \qquad x_4 = \frac{-\,k_1 B + k_1^2\sigma_1}{k_1^2 - 1},$$

where

$$2A \equiv \frac{1}{a^3} - 3a - \frac{\sigma_2}{a}$$

and B is the conjugate of A. The points $-\,A$, $-\,B$, A, and B are readily shown to be the cusps of the parastroid

$$(6) \qquad \tau^4 - x\tau^3 + \sigma_2\tau^2 - y\tau + 1 = 0,$$

where σ_2 has the value given above.

By the substitution of $1/k^2$ for k^2 in equation (3), we obtain the equation of the parastroid which results from the projection of τ at an angle, the supplement of the angle of projection used above. Making this substitution in the values of the cusps (6), we find the cusps of the new parastroid, x_1', x_2', x_3', and x_4', each of which when added to its correspondent in (6) gives σ_1. Therefore, *the two parastroids are symmetrical with respect to* $\sigma_1/2$.

If in (6) k_1 is allowed to vary, then x_1 traces out the curve

$$(8) \qquad x = \frac{kA + k^2\sigma_1}{k^2 - 1},$$

which is a hyperbola with center at $\sigma_1/2$. Comparing the locus of x_1 with that of x_3 we see that when k of the first, as it runs around the unit circle, becomes

$-k$, we have the third cusp; so these two cusps trace out the same curve. Similarly, the second and the fourth cusps trace out the same hyperbola

$$x = \frac{kB + k^2\sigma_1}{k^2 - 1}, \tag{9}$$

whose center is also at $\sigma_1/2$.

Therefore, *the four cusps of the parastroids of the system lie on two fixed hyperbolas which are concentric* (see figure 2).

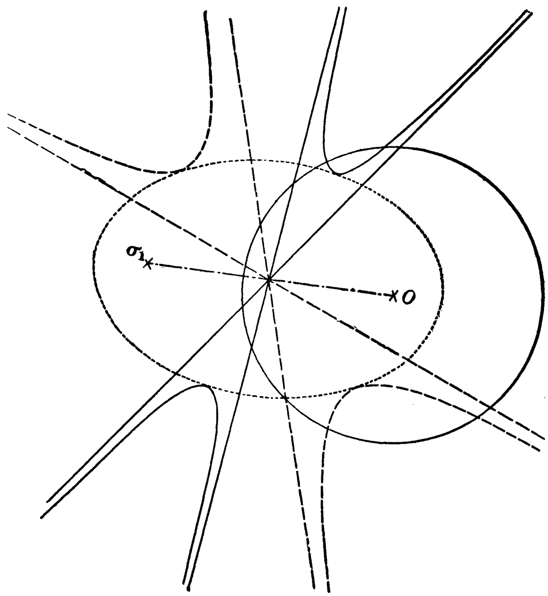

FIG. 2.

The four cusps of any parastroid of the system lie on the circle

$$[(k^2 - 1)x - \sigma_1 k^2][(k^2 - 1)y + \sigma_3] + k^2 AB = 0; \tag{10}$$

but for a varying k, this represents a system of circles whose envelope is the ellipse

$$(\sigma_3 x + \sigma_1 y - \sigma_1\sigma_3 + AB)^2 - 4AB\,xy = 0, \tag{11}$$

with center at $\sigma_1/2$, with foci at origin and σ_1, and with major axis $\sqrt{AB}$.

We can therefore say that *the cusp-circles of the system of parastroids envelop an ellipse whose center is σ_1, whose foci are the origin and σ_1 and whose major axis is equal to the absolute value of A* (see figure 2).

5. The Size, Shape, and Motion of the Parastroid. For the sake of clearness it will be well to think of the system of parastroids as if it were but a *single parastroid moving* with its center on a fixed line and changing its size according to some fixed law.

The astroid to which the parastroid (3) is parallel is

$$\tau^4 - [k^2\sigma_1 - (k^2-1)x]\,\tau^3 - [k^2\sigma_3 + k^2(k^2-1)y]\tau + k^4 = 0.$$

For any given value of τ, this equation and equation (3) represent two parallel straight lines, whose distance apart is one-half the absolute value of the difference between their constant terms, *i. e.*, $\frac{1}{2}\,|k^2\sigma_2\tau^2|$ or $\sigma_2/2$, a constant independent of both τ and k. Therefore, since the shape of a parastroid is entirely dependent on the distance of its generating line from the corresponding line of the astroid to which it is parallel, *the shape of every parastroid of our system is fixed when σ_2 is given;* or, putting it differently, *as the parastroid moves, it retains its shape* (see figure 1).

New let us notice the stroke between two adjacent cusps, say $x_1 - x_2$. From (6) we find

$$x_1 - x_2 = \frac{k(A-B)}{k^2-1}.$$

The clinant* of this stroke is

$$\frac{x_1 - x_2}{y_1 - y_2} = \frac{\dfrac{k(A-B)}{k^2-1}}{\dfrac{-k(B-A)}{k^2-1}} = 1.$$

In the same way,

$$\frac{x_1 - x_4}{y_1 - y_4} = -1.$$

From these, it is seen that the first stroke $x_1 - x_2$ is always parallel to the axis of reals, and the second $x_1 - x_4$ is always perpendicular to it; that is, *the parastroid moves parallel to itself with its four cusps forming a rectangle, a side of which is always parallel to the axis of reals* (see figure 1).

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* Franklin, "On some applications of circular coordinates," *Amer. Jour. of Math.*, vol. 12(1890), p. 162.

The area of a parastroid of constant shape varies as the square of the absolute value of a diagonal, say $x_1 - x_3$. Then from (6), we obtain

$$|x_1 - x_3|^2 = -\frac{4k^2}{(k^2-1)^2} AB,$$

as the square of a diagonal. From the expression for the center of a parastroid, we obtain as the square of the distance of this center from the origin

$$|x_0|^2 = -\frac{k^2}{(k^2-1)^2}\sigma_1\sigma_3 .$$

On comparing this with the square of a diagonal, there results a constant,

$$\frac{|x_1 - x_3|^2}{|x_0|^2} = \frac{4AB}{\sigma_1\sigma_3} .$$

Hence, *the area of the moving parastroid varies directly as the square of the distance of its center from the origin.*

6. The Cusp Loci — the two Hyperbolas. The equations of these two hyperbolas have already been given. They are

$$(8) \qquad x = \frac{kA + k^2\sigma_1}{k^2 - 1},$$

$$(9) \qquad x = \frac{kB + k^2\sigma_1}{k^2 - 1}.$$

Since both have the same form, it will be necessary to consider but one of them. The first (8) may be written

$$x = \frac{kA}{k^2-1} + \frac{k^2\sigma_1}{k^2-1},$$

which shows the curve as the resultant of two simpler curves; namely,

$$1) \qquad x_1 = \frac{kA}{k^2-1},$$

the equation of a line (counted twice) which passes through the origin, perpendicular to the stroke A, but with the part from $+\,iA/2$ to $-\,iA/2$ omitted; and

$$2) \qquad x_2 = \frac{k^2\sigma_1}{k^2-1},$$

the equation of a line perpendicular to the stroke σ_1 at its mid-point. These two equations can be easily combined, for, on giving k some particular value k_1, we have

$$|x_1| = \sqrt{\frac{-k_1^2}{(k_1^2-1)^2}AB},$$

$$|x_2| = \sqrt{\frac{-k_1^2}{(k_1^2-1)^2}\sigma_1\sigma_3},$$

whose ratio is evidently the ratio of the absolute values of the strokes A and σ_1. Thus, corresponding points on the two loci can be marked and added so as to form the points of the hyperbola.*

The asymptotes of (8) are readily found. The equation of any line through the point $\sigma_1/2$ is

$$2x + 2ty = \sigma_1 + t\sigma_3.$$

On substituting (8) and its conjugate in this equation, we obtain those values of k which give the points of intersection of the line and the hyperbola. They are the roots of the equation.

$$2(kA + \sigma_1 k^2 - tkB - t\sigma_3) = (k^2 - 1)(\sigma_1 + t\sigma_3).$$

But it is seen from (8) that, when $k = \pm 1$, x is infinite; hence these values of k give the clinants t of those lines through the center $\sigma_1/2$ which cut the hyperbola at infinity. In this way, we find

$$t_1 = \frac{\sigma_1 - A}{\sigma_3 - B}, \qquad t_2 = \frac{\sigma_1 + A}{\sigma_3 + B}.$$

Therefore the two asymptotes of the hyperbola (8) are

$$2\left(x + \frac{\sigma_1 \pm A}{\sigma_3 \pm B}\,y\right) = \sigma_1 + \frac{\sigma_1 \pm A}{\sigma_3 \pm B}\,\sigma_3, \tag{12}$$

where the upper signs give one and the lower signs the other asymptote.

In a similar manner, the asymptotes of (9) are found to be

$$2\left[x + \frac{\sigma_1 \pm B}{\sigma_3 \pm A}\,y\right] = \sigma_1 + \frac{\sigma_1 \pm B}{\sigma_3 \pm A}\,\sigma_3. \tag{13}$$

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* This gives a very simple construction of the hyperbola whose map equation has the form (8).

If lines be drawn from the point σ_1 to the two cusps A and $-A$ of (7), these strokes are $\sigma_1 - A$ and $\sigma_1 + A$, whose clinants are respectively

$$\frac{\sigma_1 - A}{\sigma_3 - B} \quad \text{and} \quad \frac{\sigma_1 + A}{\sigma_3 + B}.$$

On comparing these with the clinants of the asymptotes (12), it is seen that they differ only in sign. Therefore, *the asymptotes of the hyperbola* (8) *are perpendicular to the strokes from* σ_1 *to the two opposite cusps* A *and* $-A$ *of the parastroid* (7). (See figure 3).

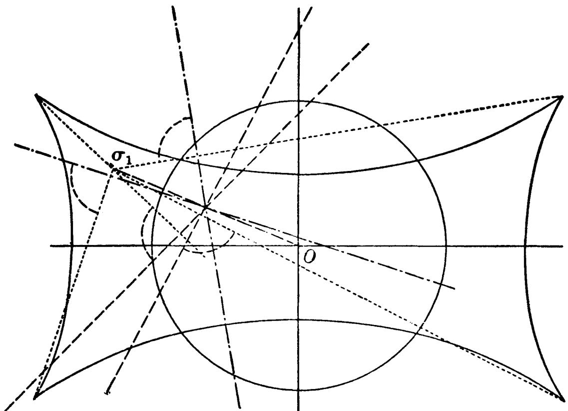

FIG. 3.

In a similar manner, it is shown that *the asymptotes of* (9) *are perpendicular to the strokes from* σ_1 *to the other two cusps* B *and* $-B$ *of* (7).*

The hyperbola (8) will be rectangular when its asymptotes are perpendicular, that is, when from (12)

$$\frac{\sigma_1 + A}{\sigma_1 + B} = -\frac{\sigma_1 - A}{\sigma_3 - B}, \quad \text{or} \quad \sigma_1\sigma_3 = AB.$$

Therefore, *if the given points* t_i *on the unit circle are such that* σ_1 *is at one of the cusps of the parastroid* (7), *the hyperbola* (8) *will be rectangular.*

* A similar theorem is true for the case of 3 fixed t's, discussed by Dr. Converse, and may be stated thus: *The three asymptotes of the cusp-locus of the deltoids of the system* C_3 *are perpendicular to the strokes from* σ_1 *to the three cusps of the deltoid*

$$t^3 - xt^2 + \sigma_3 yt - \sigma_3 = 0.$$

Since the same condition holds for (9), both hyperbolas are rectangular at the same time.

The two following theorems are easily established:

The four common tangents of the two hyperbolas form a rectangle, a side of which is parallel to the axis of reals (see figure 4).

The ellipse, enveloped by the cusp-circles of the system of parastroids,

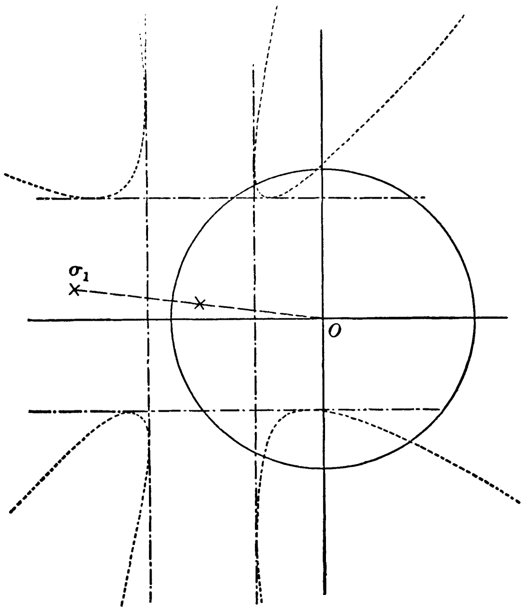

FIG. 4.

touches each of the two hyperbolas, the locus of cusps, in two points (see figure 2).

7. Special Shapes of the Parastroid. From what was said in §5, it is evident that the shape of any parastroid of the system (3) is the same as that of the parastroid (7), so here for the sake of simplicity we need only discuss the shapes of the latter, that is, the shape of

$$(7) \qquad \tau^4 - x\tau^3 + \sigma_2\tau^2 - y\tau + 1 = 0.$$

This will evidently be an astroid when $\sigma_2 = 0$, so the only condition to be imposed on the four given points t_i of the unit circle is that $\sigma_2 = 0$.

The parastroid will be tangent to itself when it passes through its own center, which in this case is the origin. Therefore, in the map equation of (7), on putting x equal to zero, we find

$$\sigma_2 = -3\tau^2 + \frac{1}{\tau^2}.$$

But since σ_2 must always be real when $\sigma_4 = 1$, it must be equal to its conjugate, so

$$\sigma_2 = -3\tau^2 + \frac{1}{\tau^2} = -\frac{3}{\tau^2} + \tau^2,$$

or

$$\tau = \pm 1.$$

On putting this value for τ above, there results

$$\sigma_2 = \pm 2,$$

which is the necessary and sufficient condition that the parastroid be tangent to itself.

The only remaining shape of a parastroid of special interest is the one whose cusps have coincided by twos, forming an ellipse-shaped curve.* Obviously the condition here is (see p. 163)

$$A = B.$$

On substituting the values of A and B in terms of σ_2, this reduces to the form

$$\sigma_2 = \pm 6,$$

which is the condition on the t's that the curve may have the desired shape.

The fixed points on the unit circle may be readily chosen in order to satisfy any one of these three conditions.

Wesleyan University,
Middletown, Conn.,
February, 1906.

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* For properties of this curve see Wolstenholme's *Mathematical Problems*, p. 303.